On the Structure of Submanifolds with Degenerate Gauss Maps

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Abstract. An n-dimensional submanifold X of a projective space $P^N(\mathbf{C})$ is called tangentially degenerate if the rank of its Gauss mapping $\gamma : X \rightarrow G(n, N)$ satisfies $0 < \operatorname{rank} \gamma < n$.

The authors systematically study the geometry of tangentially degenerate submanifolds of a projective space $P^N(\mathbf{C})$. By means of the focal images, three basic types of submanifolds are discovered: cones, tangentially degenerate hypersurfaces, and torsal submanifolds. Moreover, for tangentially degenerate submanifolds, a structural theorem is proven. By this theorem, tangentially degenerate submanifolds that do not belong to one of the basic types are foliated into submanifolds of basic types. In the proof the authors introduce irreducible, reducible, and completely reducible tangentially degenerate submanifolds. It is found that cones and tangentially degenerate hypersurfaces are irreducible, and torsal submanifolds are completely reducible while all other tangentially degenerate submanifolds not belonging to basic types are reducible.

Keywords: tangentially degenerate submanifold, submanifold with degenerate Gauss mapping, structure theorem.

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0 Introduction

An n-dimensional submanifold X of a projective space $P^N(\mathbf{C})$ is called *tangentially degenerate* if the rank of its Gauss mapping $\gamma : X \rightarrow G(n, N)$ is less than n, $r = \operatorname{rank} \gamma < n$. Here $x \in X$, $\gamma(x) = T_x(X)$, and $T_x(X)$ is the tangent subspace to X at x considered as an n-dimensional projective space P^n. The number r is also called the *rank* of X, $r = \operatorname{rank} X$. The case $r = 0$ is trivial one: it gives just an n-plane. A submanifold X is called tangentially degenerate if $0 \leq r < n$, and it is denoted by V_r^n, $X = V_r^n$. The submanifolds of rank $r < n$ have been the object of numerous investigations because of their analogy to developable surfaces in a three-dimensional space and because of their significance in the theory of the curvature of submanifolds.

The tangentially degenerate submanifolds X of rank $r < n$ were first considered by É. Cartan [C 16] in connection with his study of metric deformations of hypersurfaces, and in [C 19] in connection with his study of manifolds of constant curvature. In particular, Cartan proved that if V_r^n is a tangentially degenerate submanifold of dimension n and rank r and the dimension ρ of the osculating subspace is $\rho = n + \frac{1}{2}r(r+1)$, then V_r^n is a cone with an $(n-r-1)$-dimensional vertex.

It appeared that tangentially degenerate submanifolds are less rigid under the affine and projective deformations. Yanenko [Ya 53] studied them in connection with his study of metric deformations of submanifolds. Akivis [A 57,

62] studied them in multidimensional projective space, considered their focal images (the locus of singular points and the locus of singular hyperplanes) and applied the latter to clarify the structure of the tangentially degenerate submanifolds. Savelyev [Sa 57, 60] found a classification of tangentially degenerate submanifolds and described in detail the tangentially degenerate submanifolds of rank 2. Ryzhkov [R 58] showed that a tangentially degenerate submanifolds X of rank r can be constructed by using the Peterson transformation of r-dimensional submanifolds, and in [R 60] he proved that such a construction is quite general. That is, by means of it, an arbitrary tangentially degenerate submanifold X of rank r can be obtained (see also the survey paper [AR 64]). In particular, Ryzhkov [R 60] generalized the above mentioned Cartan's result to the case $n + 1 + \frac{1}{2}r(r-1) < \rho < n + \frac{1}{2}r(r+1)$. Brauner [Br 38], Wu [Wu 95], and Fischer and Wu [FW 95] studied such submanifolds in an Euclidean N-space.

For a submanifold V^n of a Riemannian space $V^N,\ n < N$, Chern and Kuiper [CK 52] introduced the notion of the index of relative nullity $\mu(x)$, where $x \in V^n$ (see also [KN 69], p. 348). The submanifolds V^n, for which $\mu(x)$ is constant and greater than 0 for all points $x \in V^n$, are called *strongly parabolic*. Akivis [A 87] proved that if a space V^N admits a projective realization (this is always the case for the simply connected Riemannian spaces of constant curvature, see [W 72], Ch. 2) and if the index $\mu(x)$ is constant on $V^n \subset V^N$, then the index $\mu(x)$ is connected with the rank of a submanifold V^n by the relation $\mu(x) = n - r$. This implies that the results of the papers [A 57, 62] as well as the results of the current paper can be applied to the study of strongly parabolic submanifolds of the Euclidean and non-Euclidean spaces. In particular, in [A 87] Akivis proved the existence of tangentially degenerate submanifolds in such spaces without singularities and constructed examples of such submanifolds. The main results of papers indicated above can be found in Chapter 4 of the book [AG 93]. In the same paper [A 87], Akivis also proved that a hypersurface in a four-dimensional Euclidean space $\mathbf{R}^4$ considered by Sacksteder [S 60] is of rank 2 and without singularities, and that this hypersurface is a particular case of a series of examples presented in [A 87]. Later Akivis and Goldberg [AG 00] proved that a similar example constructed (but not published) by Bourgain and published in [Wu 95], [I 98, 99a, 99b], and [WZ 99] coincides with Sacksteder's example up to a coordinate transformation.

Griffiths and Harris [GH 79] (Section 2, pp. 383–393) considered tangentially degenerate submanifolds from the point of view algebraic geometry. They used the term "submanifolds with degenerate Gauss mappings" instead of the term "tangentially degenerate varieties". They used this term to avoid a confusion with submanifolds with degenerate tangential varieties considered in Section 5 of the same paper.

Following [GH 79], Landsberg [L 96] considered tangentially degenerate submanifolds. His recently published notes [L 99] are in some sense an update to the paper [GH 79]. Section 5 (pp. 47–50) of these notes is devoted to tangentially degenerate submanifolds.

Griffiths and Harris [GH 79] asserted a structure theorem for submanifolds

with degenerate Gauss mappings, that is, for the varieties $X = V_r^n$ such that $\dim \gamma(X) < \dim X$. They asserted that such varieties are "built up from cones and developable varieties" (see [GH 79], p. 392). They gave a proof of this assertion in the case $n = 2$. However, their assertion is not completely correct. In a recent note [AGL], Akivis, Goldberg, and Landsberg present counter-examples to Griffiths–Harris' assertion when $n > 2$, and in particular, they prove that this assertion is false even for hypersurfaces with one-dimensional fibers.

Recently four papers [I 98, 99a, 99b] and [IM 97] on tangentially degenerate submanifolds (called "developable" in these papers) were published. In [IM 97], the authors found the connection between such submanifolds and solutions of Monge-Ampère equations, with the foliation of plane generators L of X called the Monge-Ampère foliation. In [IM 97], the authors proved that the rank of the Gauss map of a compact tangentially degenerate C^∞-hypersurface $X \subset \mathbf{R}P^N$ is an even integer satisfying the inequality $\frac{r(r+3)}{2} < N$, $r \neq 0$, and that if $r \leq 1$, then X is necessarily a projective hyperplane of $\mathbf{R}P^N$. If $N = 3$ or $N = 5$, then a compact tangentially degenerate C^∞-hypersurface is a projective hyperplane.

In [I 98, 99b], Ishikawa found real algebraic cubic nonsingular tangentially degenerate hypersurface in $\mathbf{R}P^N$ for $N = 4, 7, 13, 25$, and in [I 99a] he studied singularities of tangentially degenerate C^∞-hypersurfaces.

In 1997 Borisenko published the survey paper [B 97] in which he discussed results on strongly parabolic submanifolds and related questions in Riemannian and pseudo-Riemannian spaces of constant curvature and in particular, in an Euclidean space E^N. Among other results, he gives a description of certain classes of submanifolds of arbitrary codimension that are analogous to the class of parabolic surfaces in an Euclidean space E^3. Borisenko also investigates the local and global metric and topological properties; indicates conditions which imply that a submanifold of an Euclidean space E^N is cylindrical; presents results on strongly parabolic submanifolds in pseudo-Riemannian spaces of constant curvature, and finds the relationship with minimal surfaces.

In the current paper we study systematically the differential geometry of tangentially degenerate submanifolds of a projective space $P^N(\mathbf{C})$. By means of the focal images, three basic types of submanifolds are discovered: cones, tangentially degenerate hypersurfaces, and torsal submanifolds. Moreover, for tangentially degenerate submanifolds, a structural theorem is proven. By this theorem, tangentially degenerate submanifolds that do not belong to the basic types are foliated into submanifolds of basic types. In the proof we introduce irreducible, reducible, and completely reducible tangentially degenerate submanifolds. It is found that cones and tangentially degenerate hypersurfaces are irreducible and torsal submanifolds are completely reducible while all other tangentially degenerate submanifolds not belonging to basic types are reducible. Particular examples of tangentially degenerate submanifolds as well as tangentially degenerate submanifolds of low dimensions is considered in [AGL]. Some examples of tangentially degenerate submanifolds can be also found in [A 87].

In this paper we apply the method of exterior forms and moving frames of Cartan [C 45] which was often successfully used in differential geometry.

1 The main results

Let $X \subset P^N(\mathbf{C})$ be an n-dimensional smooth submanifold with a degenerate Gauss map $\gamma : X \to G(n, N)$, $\gamma(x) = T_x(X)$, $x \in X$. Suppose that $\operatorname{rank} \gamma = r < n$. Denote by L a leave of this map, $L = \gamma^{-1}(T_x) \subset X$.

Theorem 1 *A leave L of the Gauss map γ is a subspace of P^N, $\dim L = n - r = l$ or its open part.*

For the proof of this theorem see [AG 93] (p. 115, Theorem 4.1).

The foliation on X with leaves L is called the *Monge-Ampère foliation* (see, for example, [D 89] or [I 98, 99b]). In this paper, we extend the leaves of the Monge-Ampère foliation to a projective space P^l assuming that $L \sim P^l$ is a plane generator of the submanifold X. As a result, we have $X = f(P^l \times M^r)$, where M^r is a parametric variety, and f is a differentiable map $f : P^l \times M^r \to P^N$.

However, unlike a traditional definition of the foliation (see for example, [DFN 85], §29), the leaves of the Monge-Ampère foliation have singularities. This is a reason that in general its leaves are not diffeomorphic to a standard leaf.

The tangent subspace $T_x(X)$ is fixed when a point x moves along L. This is the reason that we denote it by T_L, $L \subset T_L$. A pair (L, T_L) on X depends on r parameters.

With a second-order neighborhood of a pair (L, T_L), two systems of square $r \times r$ matrices $B^\alpha = (b^\alpha_{pq})$ and $C_i = (c^q_{pi})$, $i = 0, 1, \ldots, l$; $p, q = l+1, \ldots, n$; $\alpha = n+1, \ldots, N$, are associated. The equation $\det(C_i x^i) = 0$ defines the set of singular points $x = (x^i) \in L$ of the map γ. This set is an algebraic hypersurface $F_L \subset L$ of degree r which is called the *focus hypersurface*. The equation $\det(\xi_\alpha B^\alpha) = 0$ defines the set of singular tangent hyperplanes $\xi = (\xi_\alpha) \supset T_L$ of the map γ. This set is an algebraic hypercone Φ_L with vertex T_L which is called the *focus hypercone*.

It appears that the products $H^\alpha_i = B^\alpha C_i = (h^\alpha_{ipq})$ are symmetric. They define on X the second fundamental forms $h^\alpha_i = h^\alpha_{ipq} \theta^p \theta^q$, where θ^p are basis forms of the manifold M. We assume that *for all values of parameters $u = (u^p) \in M$, the system of forms h^α_i is regular*, i.e., among them there is at least one nondegenerate quadratic form.

Denote by S_L the osculating subspace of X which is constant at all points $x \in L$ of its generator L. Its dimension is $\dim S_L = n + m$, where m is the number of linearly independent second fundamental forms of X.

We assume that the conditions of all theorems in this paper are satisfied for all values of parameters $u \in M$.

Theorem 2 *Suppose that $l \geq 1$ and $m \geq 2$, and the focus hypersurfaces F_L and the focus hypercones Φ_L do not have multiple components. Then the submanifolds X is foliated into r families of torses with l-dimensional plane generators L. Each of these families depends on $r - 1$ parameters.*

For the definition of torse see Example 2 in Section 2. A manifold X described in Theorem 2 is called *torsal*.

Theorem 3 *Suppose that $l \geq 2$, and the focus hypersurfaces F_L do not have multiple components and are indecomposable. Then the submanifold X is a hypersurface of rank r in a subspace $P^{n+1} \subset P^N$.*

Theorem 4 *Suppose that $m \geq 2$, and the focus hypercones Φ_L do not have multiple components and are indecomposable. Then the submanifold X is a cone with an $(l-1)$-dimensional vertex and l-dimensional plane generators.*

The system of matrices B^α and C_i associated with a submanifold X is said to be *reducible* if these matrices can be simultaneously reduced to a block diagonal form:

$$C_i = \operatorname{diag}\,(C_{i1}, \ldots, C_{is}), \quad B^\alpha = \operatorname{diag}\,(B_1^\alpha, \ldots, B_s^\alpha), \tag{1}$$

where C_{it} and B_t^α, $t = 1, \ldots, s$, are square matrices of orders r_t, and $r_1 + r_2 + \ldots + r_s = r$. If such a decomposition of matrices is not possible, the system of matrices B^α and C_i is called *irreducible*. If $r_1 = r_2 = \ldots = r_s = 1$, then the system of matrices B^α and C_i is called *completely reducible*.

A manifold X with a degenerate Gauss mapping is said to be *reducible, irreducible* or *completely reducible* if for any values of parameters $u \in M$ the matrices B^α and C_i are reducible, irreducible or completely reducible, respectively.

Theorem 5 *Suppose that a manifold X is reducible, and its matrices B_t^α and C_{it} defined in* (1) *are of order r_t, $t = 1, \ldots, s$. Then X is foliated into s families of $(l + r_t)$-dimensional submanifolds of rank r_t with l-dimensional plane generators. For $r_t = 1$, these submanifolds are torses, and for $r_t \geq 2$, they are irreducible submanifolds described in Theorems* 3 *and* 4.

2 Examples of submanifolds with degenerate Gauss maps

Consider a few examples of submanifolds with a degenerate Gauss map.

Example 1 For $r = 0$, a submanifold X is an n-dimensional subspace P^n, $n < N$. This submanifold is the only tangentially degenerate submanifold without singularities in P^N.

Example 2 Let Y be a curve of class C^p in the space P^N. Suppose that P^N is a space of minimal dimension containing the curve Y. Denote by L_y the osculating subspace of order l, $l \leq p$, $l \leq N-1$, of the curve Y at a point $y \in Y$. Since $\dim L_y = l$, it follows that when a point y moves along the curve Y, the subspace L_y sweeps a submanifold X of dimension $n = l + 1$ in the space P^N. At any point $x \in L_y$, the tangent subspace $T_x(X)$ coincides with the osculating subspace L'_y of order $l+1$ of the curve Y, $\dim L'_y = l + 1$, and the focus hypersurface in L_y is the osculating subspace $'L_y$ of order $l-1$ and dimension $l-1$ of the curve Y. Thus $\dim X = l + 1$, and the manifold X is

tangentially degenerate of rank $r = 1$. Such a manifold X is called a *torse*. Conversely, a submanifold of dimension n and rank 1 is a torse formed by a family of osculating subspaces of order $n-1$ of a curve of class C^p, $p \geq n-1$, in the space P^N.

In what follows, unless otherwise stated, we always assume that $r > 1$.

Example 3 Suppose that S is a subspace of the space P^N, $\dim S = l-1$, and T is its complementary subspace, dim $T = N - l$, $T \cap S = \emptyset$. Let Y be a smooth tangentially nondegenerate submanifold of the subspace T, $\dim Y =$ rank $Y = r < N - l$. Consider an r-parameter family of l-dimensional subspaces $L_y = S \wedge y$, $y \in Y$. This manifold is a cone X with vertex S and the director manifold Y. The subspace $T_x(X)$ tangent to the cone X at a point $x \in L_y$ is defined by its vertex S and the subspace $T_y(Y)$, $T_x(X) = S \wedge T_y(X)$, and $T_x(X)$ remains fixed when a point x moves in the subspace L_y. As a result, the cone X is a tangentially degenerate submanifold of dimension $n = l + r$ and rank r, with plane generators L_y of dimension l.

Example 4 Let T be a subspace of dimension $n+1$ in P^N, and let Y be an r-parameter family of hyperplanes ξ in general position in T, $r < n$. Such a family has an n-dimensional envelope X that is a tangentially degenerate submanifold of dimension n and rank r in the subspace T. It is foliated into an r-parameter family of plane generators L of dimension $l = n - r$ along which the tangent subspace $T_x(X)$, $x \in L$, is fixed and coincides with a hyperplane ξ of the family in question. Thus X is a tangentially degenerate hypersurface of rank r with $(n-r)$-dimensional flat generators L in the space T.

Example 5 If $\operatorname{rank} X = \dim X = n$, then X is a submanifold of complete rank. X is also called a *tangentially nondegenerate* in the space P^N.

3 Application of the duality principle

By the duality principle, to a point x of a projective space P^N, there corresponds a hyperplane ξ. A set of hyperplanes of space P^N forms the dual projective space $(P^N)^*$ of the same dimension N. Under this correspondence, to a subspace P of dimension p, there corresponds a subspace $P^* \subset (P^N)^*$ of dimension $N-p-1$. Under the dual map, the incidence of subspaces is preserved, that is, if $P_1 \subset P_2$, then $P_1^* \supset P_2^*$.

In the space P^N, if a point x describes a tangentially nondegenerate manifold X, $\dim X = r$, then in general, the hyperplane ξ corresponding to x envelopes a hypersurface X^* of rank r with $(N-r-1)$-dimensional generators. The dual map sends a smooth manifold $X \subset P^N$ of dimension n and rank r with plane generators L of dimension $l = n-r$ to a manifold X^* of dimension $n^* = N-l-1$ and the same rank r with plane generators L^* of dimension $l^* = N-n-1$. Under

this map, to a tangent subspace $T_x(X)$ of the submanifold X there corresponds the plane generator L^*, and to a plane generator L there corresponds the tangent subspace of the manifold X^*.

Note that Ein [E 85, 86] applied the duality principle for studying tangentially degenerate varieties with small dual varieties.

Let us determine which manifolds correspond to tangentially degenerate manifolds considered in Examples 2–4. To a cone X of rank r with vertex S of dimension $l-1$ (see Example 3), there corresponds a manifold X^* lying in the subspace $T = S^*$, $\dim T = N - l$. Since $\dim X^* = n^* = N - l - 1$, the manifold X^* is a hypersurface of rank r in the subspace T. Such a hypersurface was considered in Example 4. It follows that Examples 3 and 4 are mutually dual one to another.

Suppose that the subspace T containing a hypersurface X of rank r coincides with the space P^N. Then under the dual map in T, to the hypersurface X there corresponds an r-dimensional tangentially nondegenerate manifold X^*.

Under the dual map, in P^N to torses (see Example 2) there correspond torses enveloping a one-parameter family of hyperplanes η—the images of points of the curve Y. These torses are of dimension $N-1$ and rank 1, and their plane generators are of dimension $l = N-2$.

Note that the dual map allows us to construct tangentially degenerate manifolds in the space P^{n+1} from the general tangentially nondegenerate manifolds.

Example 6 Consider a set of conics in a projective plane P^2. They are defined by the equation

$$a_{ij}x^i x^j = 0, \;\; i, j = 0, 1, 2, \tag{2}$$

where the coefficients a_{ij} are defined up to a constant factor, and $a_{ij} = a_{ji}$. Thus to the curve (2) there corresponds a point of a five-dimensional projective space P^{5*} whose coordinates coincide with the coefficients of equation (2).

Among the conics defined by equation (2) there are singular curves which decompose into two straight lines. Such conics are distinguished by the condition

$$\det(a_{ij}) = 0. \tag{3}$$

Equation (3) is of degree three and defines a hypersurface in P^{5*} which is called the *cubic symmetroid.*

We prove now that *the hypersurface defined by equation* (3) *is tangentially degenerate of rank two and bears a two-parameter family of two-dimensional plane generators.* To this end, we write equation (3) in the form

$$F = \det \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} = 0. \tag{4}$$

The equation of tangent subspaces of the hypersurface (4) has the form

$$\xi^{ij} a_{ij} = 0,$$

where $\xi^{ij} = \dfrac{\partial F}{\partial a_{ij}}$, $\xi^{ij} = \xi^{ji}$, are the cofactors of the entries a_{ij} in the determinant in equation (4). The quantities ξ^{ij} are coordinates in the projective space P^5, which is dual to the space P^{5*} where we defined the cubic symmetroid (4).

Consider the symmetric matrix $\xi = (\xi^{ij})$. A straightforward computation shows that the rank of this matrix is equal to one. Thus the entries of the matrix ξ can be represented in the form

$$\xi^{ij} = \xi^i \xi^j, \tag{5}$$

where ξ^i are projective coordinates of a point in the projective plane P^2. But equations (5) define the Veronese surface in the space P^5. Since the Veronese surface is of dimension two, its points depend on two affine parameters $u = \frac{\xi^1}{\xi^0}$ and $v = \frac{\xi^2}{\xi^0}$. As a result, the tangent hyperplanes of the cubic symmetroid (4) depend on two parameters, and this hypersurface is of rank two.

To the plane generators of the hypersurface (4), there correspond two-parameter families of conics in P^2 decomposing into pairs of straight lines with a common intersection point.

Thus, *the cubic symmetroid* (4) *and the Veronese surface* (5) *are mutually dual submanifolds of a five-dimensional projective space.*

4 Basic equations and focal images

We study tangentially degenerate submanifolds applying the method of moving frames in a projective space P^N. In P^N, we consider a manifold of projective frames $\{A_0, A_1, \ldots, A_N\}$. On this manifold

$$dA_u = \omega_u^v A_v, \;\; u, v, = 0, 1, \ldots, N, \tag{6}$$

where the sum $\omega_u^u = 0$. The 1-forms ω_u^v are linearly expressed in terms of the differentials of parameters of the group of projective transformations of the space P^N. The total number of these parameters is $N^2 - 1$. These 1-forms satisfy the structure equations

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v \tag{7}$$

of the space P^N (see, for example, [AG 93], p. 19). Equations (7) are the conditions of complete integrability of equations (6).

Consider a tangentially degenerate submanifold $X \subset P^N$, $\dim X = n$, rank $X = r \leq n$. In addition, as above, let L be a rectilinear generator of the manifold X, $\dim L = l$; let T_L, $\dim T_L = n$, be the tangent subspace to X along the generator L, and let M be a base manifold for X, $\dim M = r$. Denote by θ^p, $p = l+1, \ldots, n$, basis forms on the variety M. These forms satisfy the structure equations

$$d\theta^p = \theta^q \wedge \theta_q^p, \;\; p, q = l+1, \ldots, n, \tag{8}$$

of the variety M. Here θ_q^p are 1-forms defining transformations of first-order frames on M.

For a point $x \in L$, we have $dx \in T_L$. With X, we associate a bundle of projective frames $\{A_i, A_p, A_\alpha\}$ such that $A_i \in L,\ i = 0, 1, \ldots, l;\ A_p \in T_L,\ p = l+1, \ldots, n$. Then

$$\begin{cases} dA_i = \omega_i^j A_j + \omega_i^p A_p, \\ dA_p = \omega_p^i A_i + \omega_p^q A_q + \omega_p^\alpha A_\alpha,\ \alpha = n+1, \ldots, N. \end{cases} \tag{9}$$

It follows from the first equation of (9) that

$$\omega_i^\alpha = 0. \tag{10}$$

Since for $\theta^p = 0$ the subspaces L and T_L are fixed, we have

$$\omega_i^p = c_{qi}^p \theta^q,\ \ \omega_p^\alpha = b_{qp}^\alpha \theta^q. \tag{11}$$

Since the manifold of leaves $L \subset X$ depends on r essential parameters, the rank of the system of 1-forms ω_i^p is equal to r, rank $(\omega_i^p) = r$. Similarly, we have rank $(\omega_p^\alpha) = r$.

Denote by C_i and B^α the $r \times r$ matrices occurring in equations (11):

$$C_i = (c_{qi}^p),\ \ B^\alpha = (b_{qp}^\alpha).$$

These matrices are defined in a second-order neighborhood of the submanifold X.

Exterior differentiation of equations (10) by means of structure equations (7) leads to the exterior quadratic equations

$$\omega_p^\alpha \wedge \omega_i^p = 0.$$

Substituting expansions (11) into the last equations, we find that

$$b_{qs}^\alpha c_{pi}^s = b_{ps}^\alpha c_{qi}^s. \tag{12}$$

Equations (10), (11), and (12) are called the *basic equations* in the theory of tangentially degenerate submanifolds.

Relations (12) can be written in the matrix form

$$(B^\alpha C_i)^T = (B^\alpha C_i),$$

i.e., the matrices

$$H_i^\alpha = B^\alpha C_i = (b_{qs}^\alpha c_{pi}^s)$$

are symmetric.

Let $x = x^i A_i$ be an arbitrary point of a leaf L. For such a point we have

$$dx = (dx^i + x^j \omega_j^i) A_i + x^i \omega_i^p A_p.$$

It follows that

$$dx \equiv (A_p c^p_{qi} x^i)\theta^q \pmod{L}.$$

The tangent subspace T_x to the manifold X at a point x is defined by the points A_i and

$$\widetilde{A}_q(x) = A_p c^p_{qi} x^i,$$

and therefore $T_x \subset T_L$.

A point x is a *regular* point of a leaf L if $T_x = T_L$. Regular points are determined by the condition

$$J(x) = \det(c^q_{pi} x^i) \neq 0. \tag{13}$$

If $J(x) = 0$ at a point x, then T_x is a proper subspace of T_L, and a point x is said to be a *singular point* of a leaf L.

The determinant (13) is the Jacobian of the map $f : P^l \times M^r \to P^N$. Singular points of a leaf L are determined by the condition $J(x) = 0$. In a leaf L, they form an algebraic submanifold of dimension $l - 1$ and degree r. This hypersurface (in L) is called the *focus hypersurface* and is denoted by F_L. By (13), the equations $J(x) = 0$ of the focus hypersurface on the plane generator L of the manifold X can be written as

$$\det(c^q_{pi} x^i) = 0. \tag{14}$$

We calculate now the second differential of a point $x \in L$:

$$d^2 x \equiv A_\alpha \omega^\alpha_s \omega^s_i x^i \pmod{T_x}.$$

This expression is the *second fundamental quadratic form* of the manifold X:

$$II_x = A_\alpha \omega^\alpha_s \omega^s_i x^i = A_\alpha b^\alpha_{ps} c^s_{qi} x^i \theta^p \theta^q. \tag{15}$$

Suppose that $\xi = \xi_\alpha x^\alpha = 0$ is the tangent hyperplane to X at $x \in L$, $\xi \supset T_L$. Then

$$(\xi, II_x) = h_{pq}(\xi, x)\theta^p \theta^q,$$

where

$$h_{pq}(\xi, x) = \xi_\alpha b^\alpha_{ps} c^s_{qi} x^i, \quad h_{pq} = h_{qp},$$

is the *second fundamental quadratic form* of the manifold X at x with respect to the hyperplane ξ. Since at regular points $x \in L$ condition (13) holds, the rank of this matrix is the same as the rank of the matrix

$$B(\xi) = (\xi_\alpha b^\alpha_{pq}) = \xi_\alpha B^\alpha, \tag{16}$$

and this rank is the same at all regular points $x \in L$.

We call a tangent hyperplane ξ *singular* if

$$\det(\xi_\alpha b^\alpha_{pq}) = 0, \tag{17}$$

i.e., if the rank of the matrix (16) is reduced. Condition (17) is an equation of degree r with respect to the tangential coordinates ξ_α of the hyperplane ξ. This condition defines an algebraic hypercone whose vertex is the tangent subspace T_L. This hypercone is called the *focus hypercone* and denoted by Φ_L (see [AG 93], p. 119).

The determinant $\det(\xi_\alpha b^\alpha_{pq})$ in the left-hand side of equation (17) is the Jacobian of the dual map $f^* : L^* \times M^r \to (P^N)^*,\ f^*(L^* \times M^r) = X^* \subset (P^N)^*$, where X^* is a manifold dual to X and L^* is a bundle of hyperplanes of the space P^N passing through the tangent subspace T_L of the manifold X, $\dim L^* = N - n - 1$.

The focus hypersurface $F_L \subset L$ and the focus hypercone Φ_L with vertex T_L are called *focal images* of the manifold X with a degenerate Gauss map.

Note that under the passage from the manifold $X \subset P^N$ to its dual manifold $X^* \subset (P^N)^*$, the systems of square matrices C_i and B^α as well as the focus hypersurfaces F_L and the focus cones Φ_L exchange their roles.

Since

$$d^2x \equiv A_\alpha b^\alpha_{qs} c^s_{pi} x^i \theta^p \theta^q \pmod{T_L, x \in L},$$

the points

$$A_{pq} = A_\alpha b^\alpha_{qs} c^s_{pi} x^i, \ \ A_{pq} = A_{qp}, \tag{18}$$

together with the points A_i and A_p define the osculating subspace $T^2_L(X)$. Its dimension is

$$\dim T^2_L(X) = n + m,$$

where m is the number of linearly independent points among the points $A_{pq},\ m \leq \frac{r(r+1)}{2}$. But since at a regular point $x \in L$ condition (13) holds, the number m is the number of linearly independent points among the points

$$\widetilde{A}_{pq} = A_\alpha b^\alpha_{pq}.$$

We also use the notation S_L for the osculating space $T^2_L(X)$.

On a generator L of the manifold X, consider the system of equations

$$c^q_{pi} x^i = 0. \tag{19}$$

Its matrix $C = (c^q_{pi})$ has r^2 rows and $l + 1$ columns. Denote the rank of this matrix by m^*. If $m^* < l + 1$, then system (19) defines a subspace K_L of dimension $k = l - m^*$ in L. This subspace belongs to the focus hypersurface F_L defined by equation (14). If $l > m^*$, then the hypersurface F_L becomes a cone with vertex K_L. We call the subspace K_L the *characteristic subspace* of the generator L.

Note also that by the duality principle in P^N, the osculating subspace S_L and the characteristic subspace K_L constructed for a pair (L, T_L) correspond one to another.

5 Proof of Theorem 2

Lemma 6 *Suppose that $l \geq 1$, and the focus hypersurface $F_L \subset L$ does not have multiple components. Then all matrices B^α can be simultaneously diagonalized, $B^\alpha = (\mathrm{diag}\ b^\alpha_{pp})$, and the focus hypercone Φ_L decomposes into r bundles of hyperplanes Φ_p in P^N whose axes are $(n+1)$-planes $T_L \wedge B_p$, where $B_p = b^\alpha_{pp} A_\alpha$ are points located outside of the tangent subspace T_L. The dimension $n+m$ of the osculating subspace S_L of the manifold X along a generator L does not exceed $n+r$.*

Proof. Since the hypersurface $F_L \subset L$ does not have multiple components, a general straight line λ lying in L intersects F_L at p distinct points. We place the vertices A_0 and A_1 of our moving frame onto the line λ. By (14), the coordinates of the common points of λ and F_L are defined by the equation

$$\det(c^q_{p0} x^0 + c^q_{p1} x^1) = 0.$$

Suppose that $A_0 \notin F_L$. Then $\det\ (c^q_{p0}) \neq 0$, and it is easy to prove that the matrices C_0 and C_1 can be simultaneously diagonalized, $C_0 = (\delta^p_q)$ and $C_1 = (\mathrm{diag}\ c^p_{p1})$. Since the common points of λ and F are not multiple points, we have $c^p_{p1} \neq c^q_{q1}$ for $p \neq q$.

Next we write equations (12) for $i = 0, 1$:

$$b^\alpha_{pq} = b^\alpha_{qp}, \quad b^\alpha_{qp} c^p_{p1} = b^\alpha_{pq} c^q_{q1}.$$

Since $c^p_{p1} \neq c^q_{q1}$, it follows that $b^\alpha_{pq} = 0$ for $p \neq q$. As a result, all matrices B^α can be simultaneously diagonalized, $B^\alpha = (\mathrm{diag}\ b^\alpha_{pp})$. Equation (17) takes the form

$$\prod_p (\xi_\alpha b^\alpha_{pp}) = 0,$$

and the focus hypercone Φ_L decomposes into r bundles of hyperplanes Φ_p in P^N whose axes are $(n+1)$-planes $T_L \wedge B_p$, where $B_p = b^\alpha_{pp} A_\alpha$ are points located outside of the tangent subspace T_L. The osculating subspace S_L of the manifold X along a generator L is the span of the tangent subspace T_L and the points $B_{l+1}, \ldots, B_n$. Thus, the dimension of this subspace does not exceed $n+r$. ■

Lemma 7 *Suppose that $l \geq 1$, $m \geq 2$, the focus hypersurfaces $F_L \subset L$ do not have multiple components, and all the bundles Φ_p into which the hypercone Φ_L decomposes are of multiplicity one. Then the focus hypersurface F_L decomposes into r hyperplanes $F_p \subset L$.*

Proof. Consider the matrix

$$B = (b^\alpha_{pp}) \tag{20}$$

composed from the eigenvalues of the matrices B^α. Matrix (20) has r columns and m independent rows, $m \leq r$. Write equations (12) for the diagonal matrices B^α:

$$b^\alpha_{qq} c^q_{pi} = b^\alpha_{pp} c^p_{qi}. \tag{21}$$

Since the matrix B has m linearly independent columns and $m \geq 2$, it follows from equation (21) that $c^q_{pi} = 0$ for $p \neq q$, and all the matrices C_i and B^α can be simultaneously diagonalized. Now the equation of the focus hypersurface F_L takes the form

$$\prod_{p=l+1}^{n} c^p_{pi} x^i = 0,$$

and the hypersurface F_L decomposes into r hyperplanes F_p defined in L by the equation $c^p_{pi} x^i = 0$. Since by the conditions of Theorem 2, the focus hypersurface F_L does not have multiple components, all hyperplanes F_p are distinct. ■

Consider a rectangular $r \times (l+1)$ matrix

$$C = (c^p_{pi}) \tag{22}$$

formed by the eigenvalues of the matrix C_i.

Proof of Theorem 2. From the conditions of Theorem 2 and Lemmas 6 and 7, it follows that the matrices C_i and B^α can be simultaneously diagonalized:

$$C_i = \operatorname{diag}\,(c^{l+1}_{l+1,i}, \ldots, c^n_{ni}), \;\; B^\alpha = \operatorname{diag}\,(b^\alpha_{l+1,l+1}, \ldots, b^\alpha_{nn}).$$

This implies that formulas (11) take the form:

$$\omega^p_i = c^p_{pi} \theta^p, \;\; \omega^\alpha_p = b^\alpha_{pp} \theta^p, \tag{23}$$

where θ^p are basis forms on the parametric variety M, and there is no summation over the index p. Exterior differentiation of equations (23) leads to the following exterior quadratic equations:

$$\nabla c^p_{pi} \wedge \theta^p + c^p_{pi} d\theta^p + \sum_{q \neq p} c^q_{qi} \omega^p_q \wedge \theta^q = 0, \tag{24}$$

$$\nabla b^\alpha_{pp} \wedge \theta^p + b^\alpha_{pp} d\theta^p - \sum_{q \neq p} b^\alpha_{qq} \omega^q_p \wedge \theta^q = 0, \tag{25}$$

where

$$\nabla c^p_{pi} = dc^p_{pi} - c^p_{pj} \omega^j_i + c^p_{pi} \omega^p_p, \;\; \nabla b^\alpha_{pp} = db^\alpha_{pp} + b^\beta_{pp} \omega^\alpha_\beta - b^\alpha_{pp} \omega^p_p.$$

Suppose now that at least one of the matrices B and C defined by equations (20) and (22) has mutually linearly independent columns. Then it follows from equations (24) and (25) that

$$\begin{cases} d\theta^p \equiv 0 \pmod{\theta^p}, \\ \omega^p_q \wedge \theta^q \equiv 0 \pmod{\theta^p}, \\ \omega^q_p \wedge \theta^q \equiv 0 \pmod{\theta^p}. \end{cases}$$

This implies that

$$d\theta^p = \theta^p \wedge \theta^p_p, \;\; \omega^q_p = l^q_{pp} \theta^p + l^q_{pq} \theta^q. \tag{26}$$

We write the expression of dA_p from (9) in more detail:

$$dA_p = \omega_p^i A_i + \omega_p^p A_p + \sum_{q \neq p} \omega_p^q A_q + \omega_p^\alpha A_\alpha. \tag{27}$$

Let the index p be fixed and $q \neq p$; $p, q = l{+}1, \ldots, n$. Consider a submanifold $X_p \subset X$ defined by the equations

$$\theta^q = 0, \ \ q \neq p. \tag{28}$$

The first equation of (26) implies that $\dim X_p = 1$. By (28), (9) and (27), we find that

$$dA_i = \omega_i^j A_j + c_{pi}^p \theta^p A_p, \tag{29}$$

$$dA_p = \omega_p^i A_i + \omega_p^p A_p + B_p \theta^p, \tag{30}$$

where $B_p = \sum_{q \neq p} l_{pp}^q A_q + b_{pp}^\alpha A_\alpha$. Since the points A_i are basis points of a generator L of the manifold X, equations (29) and (30) prove that the subspace $L \wedge A_p$ is tangent to the submanifold X_p at all regular points of the generator L, and the subspace $L \wedge A_p \wedge B_p$ is the osculating subspace of X_p. Singular points of a generator L of the manifold X_p are determined by the equations $c_{pi}^p x^i = 0$ and form a hyperplane in L (see Example 2 in Section 2). Thus, the submanifold X_p is a torse with l-dimensional plane generators. Hence the manifold X is foliated into r families of torses, each of these families depends on $r-1$ parameters u^q, and the forms θ^q are expressed in terms of the differentials of these parameters, $\theta^q = l^q du^q$. ■

6 Proof of Theorems 3 and 4

Lemma 8 *Suppose that $l \geq 2$, the focus hypersurfaces $F_L \subset L$ do not have multiple components and are indecomposable. Then the hypercone Φ_L is an r-multiple bundle of hyperplanes with an $(n+1)$-dimensional vertex in P^N.*

Proof. From the conditions of the lemma and equation (21) it follows that the columns of the matrix B are linearly dependent, and thus all points $B_p = b_{pp}^\alpha A_\alpha$ defined by the columns of this matrix lie in a subspace of dimension $n{+}1$ defined by the tangent subspace T_L and one of the points B_p.

Therefore all bundles of hyperplanes Φ_p into which the hypercone Φ_L decomposes coincide. ■

We formulate the lemma which is dual to Lemma 8.

Lemma 9 *Suppose that $m \geq 2$ and the focus hypercones Φ_L with their vertices T_L do not have multiple components and are indecomposable. Then the focus hypersurface $F_L \subset L$ is an r-multiple hyperplane in L.*

Proof of Theorem 3. From the conditions of Theorem 3 and Lemma 8, it follows that all pairs of columns of the matrix B are linearly dependent. Thus

all matrices B^α associated with X are mutually linearly dependent. Hence, we have

$$b^\alpha_{pq} = b^\alpha b_{pq}. \tag{31}$$

Now conditions (12) take the form

$$b_{ps}c^s_{qi} = b_{qs}c^s_{pi}. \tag{32}$$

Although the matrix $B = (b_{pq})$ is still can be diagonalized, in general, the matrices C_i do not possess this property. Thus we write equations (11) in the form

$$\omega^p_i = c^p_{qi}\theta^q, \;\; \omega^\alpha_p = b^\alpha b_{pq}\theta^q. \tag{33}$$

Exterior differentiation of equations (33) and applying structure equations (7) of the parametric variety M, we obtain the following exterior quadratic equations:

$$\nabla c^p_{qi} \wedge \theta^q = 0, \;\; (b_{pq}\nabla b^\alpha + b^\alpha \nabla b_{pq}) \wedge \theta^q = 0, \tag{34}$$

where

$$\left\{\begin{array}{l} \nabla c^p_{qi} = dc^p_{qi} - c^p_{qj}\omega^j_i + c^s_{qi}\omega^p_s - c^p_{si}\theta^s_p, \\ \nabla b^\alpha = db^\alpha + b^\beta\omega^\alpha_\beta, \\ \nabla b_{pq} = db_{pq} - b_{sq}\omega^s_p - b_{ps}\theta^s_q. \end{array}\right.$$

As we noted already in the proof of Theorem 2, if the point A_0 does not belong to the focus hypersurface F_L of a generator L, then the matrix (c^p_{q0}) is nonsingular, and by means of a frame transformation in the tangent space $T_u(M)$ to the variety M we can reduce this matrix to the form $(c^p_{q0}) = (\delta^p_q)$. As a result, equation (32) corresponding to $i = 0$ takes the form

$$b_{pq} = b_{qp}. \tag{35}$$

Hence the matrix $B = (b_{pq})$ becomes symmetric. This matrix is the matrix of the second fundamental form II of the manifold X at the point A_0. Since A_0 is a regular point of X, the matrix B is nonsingular, $\det(b_{pq}) \neq 0$.

Next, we find from the second equation of (34) that

$$\nabla b^\alpha = b^\alpha_p\theta^p, \;\; \nabla b_{pq} = b_{pqs}\theta^s.$$

Substituting these expansions into equation (34) and equating to 0 the coefficients in independent products $\theta^s \wedge \theta^q$, we find that

$$b_{pq}b^\alpha_s - b_{ps}b^\alpha_q + b^\alpha(b_{pqs} - b_{psq}) = 0.$$

Contracting these equations with the matrix $B^{-1} = (b^{pq})$ which is the inverse of B, we find that

$$(r-1)b^\alpha_s + b^\alpha(b_{pqs} - b_{psq})b^{pq} = 0.$$

Since by the theorem hypothesis $r \neq 1$, it follows that

$$b^\alpha_s = b^\alpha b_s, \tag{36}$$

where

$$b_s = \frac{1}{r-1}(b_{pqs} - b_{psq})b^{pq}.$$

Next consider equations (17) of the focus hypercone Φ_L of the manifold X. By (31), this equation becomes

$$(b^\alpha \xi_\alpha)^r \det(b_{pq}) = 0,$$

and since $\det(b_{pq}) \neq 0$, the hypercone Φ_L becomes an r-multiple bundle of hyperplanes. The axis of this bundle is a subspace of dimension $n+1$ which is the span of the tangent subspace T and the point $b^\alpha A_\alpha$.

Let us prove that if the parameters u on the variety M vary, this subspace is fixed. In fact, the basis points of the subspace T are the points A_i and A_p, and by (9) and (36), we have

$$\left\{\begin{array}{l} dA_i = \omega_i^j A_j + \omega_i^p A_p, \\ dA_p = \omega_p^i A_i + \omega_p^q A_q + b_{pq}\theta^q \cdot b^\alpha A_\alpha. \end{array}\right.$$

If we differentiate the point $b^\alpha A_\alpha$ and apply equation (36), we find that

$$d(b^\alpha A_\alpha) \equiv b_s \theta^s \cdot b^\alpha A_\alpha \pmod{A_i, A_p}.$$

This proves our last assertion.

Thus the subspace $P^{n+1} = T \wedge b^\alpha A_\alpha$ is fixed when the tangent subspace T moves along X, and X is a hypersurface in the subspace P^{n+1}. ■

Proof of Theorem 4. Theorem 4 is dual to Theorem 3 and can be proved by applying Lemma 9 in the same way as we used Lemma 8 to prove Theorem 3. ■

7 Proof of Theorem 5

Equations (14) and (17) of the focal images imply the following lemma.

Lemma 10 *If the matrices C_i and B^α of a manifold X can be reduced to the form* (1)*, then each of its focus hypersurfaces $F_L \subset L$ decomposes into s components F_t of dimension $l-1$ and degree $r_1, r_2, \ldots, r_s$, and each of its focus hypercones Φ_L decomposes into s hypercones Φ_t of the same degrees $r_1, r_2, \ldots, r_s$; $r_1 + r_2 + \ldots + r_s = r$, and with the same vertex T. In particular, if $r_1 = r_2 = \ldots = r_s = 1$, then a focus hypersurface F_L decomposes into r hyperplanes, and a focus hypercone Φ_L decomposes into r bundles of hyperplanes with* $(n+1)$*-dimensional axes.*

Proof of Theorem 5.

We prove Theorem 5 assuming that the index t takes only two values, $t = 1, 2,\ r = r_1 + r_2$, and the indices p and q have the following values:

$$p_1, q_1 = l+1, \ldots, l+r_1, \ \ p_2, q_2 = l+r_1+1, \ldots, n.$$

Then equations (11) become

$$\begin{cases} \omega_i^{p_1} = c_{q_1 i}^{p_1} \theta^{q_1}, & \omega_{p_1}^{\alpha} = b_{p_1 q_1}^{\alpha} \theta^{q_1}, \\ \omega_i^{p_2} = c_{q_2 i}^{p_2} \theta^{q_2}, & \omega_{p_2}^{\alpha} = b_{p_2 q_2}^{\alpha} \theta^{q_2}. \end{cases} \tag{37}$$

Exterior differentiation of equations (37) gives

$$\nabla c_{q_1 i}^{p_1} \wedge \theta^{q_1} + (c_{q_2 i}^{s_2} \omega_{s_2}^{p_1} - c_{s_1 i}^{p_1} \theta_{q_2}^{s_1}) \wedge \theta^{q_2} = 0, \tag{38}$$

$$\nabla b_{p_1 q_1}^{\alpha} \wedge \theta^{q_1} - (b_{s_2 q_2}^{\alpha} \omega_{p_1}^{s_2} + b_{p_1 s_1}^{\alpha} \theta_{q_2}^{s_1} \wedge \theta^{q_2} = 0, \tag{39}$$

$$\nabla c_{q_2 i}^{p_2} \wedge \theta^{q_2} + (c_{q_1 i}^{s_1} \omega_{s_1}^{p_2} - c_{s_2 i}^{p_2} \theta_{q_1}^{s_2}) \wedge \theta^{q_1} = 0, \tag{40}$$

$$\nabla b_{p_2 q_2}^{\alpha} \wedge \theta^{q_2} - (b_{s_1 q_1}^{\alpha} \omega_{p_2}^{s_1} + b_{p_2 s_2}^{\alpha} \theta_{q_1}^{s_2} \wedge \theta^{q_1} = 0, \tag{41}$$

where

$$\begin{cases} \nabla c_{q_1 i}^{p_1} = dc_{q_1 i}^{p_1} - c_{q_1 j}^{p_1} \omega_i^j + c_{q_1 i}^{s_1} \omega_{s_1}^{p_1} - c_{s_1 i}^{p_1} \theta_{q_1}^{s_1}, \\ \nabla b_{p_1 q_1}^{\alpha} = db_{p_1 q_1}^{\alpha} + b_{p_1 q_1}^{\beta} \omega_{\beta}^{\alpha} - b_{s_1 q_1}^{\alpha} \omega_{p_1}^{s_1} - b_{p_1 s_1}^{\alpha} \theta_{s_1}^{q_1}, \\ \nabla c_{q_2 i}^{p_2} = dc_{q_2 i}^{p_2} - c_{q_2 j}^{p_2} \omega_i^j + c_{q_2 i}^{s_2} \omega_{s_2}^{p_2} - c_{s_2 i}^{p_2} \theta_{q_2}^{s_2}, \\ \nabla b_{p_2 q_2}^{\alpha} = db_{p_2 q_2}^{\alpha} + b_{s_2 q_2}^{\beta} \omega_{\beta}^{\alpha} - b_{s_2 q_2}^{\alpha} \omega_{p_2}^{s_2} - b_{p_2 s_2}^{\alpha} \theta_{q_2}^{s_2}. \end{cases}$$

Consider the system of equations

$$\theta^{q_1} = 0 \tag{42}$$

on the manifold X. By (8), its exterior differentiation gives

$$\theta^{q_2} \wedge \theta_{q_2}^{q_1} = 0. \tag{43}$$

It follows from (43) that the conditions of complete integrability of equations (42) have the form

$$\theta_{q_2}^{q_1} = l_{q_2 s_2}^{q_1} \theta^{s_2}, \tag{44}$$

where $l_{q_2 s_2}^{q_1} = l_{s_2 q_2}^{q_1}$.

By equations (42), the system of equations (38) takes the form

$$(c_{q_2 i}^{s_2} \omega_{s_2}^{p_1} - c_{s_1 i}^{p_1} \theta_{q_2}^{s_1}) \wedge \theta^{q_2} = 0. \tag{45}$$

Suppose that the component F_1 of the focus hypersurface F_L does not have multiple components. Assuming that $l \geq 1$, we write equations (45) for two different values of the index i, for example, for $i = 0, 1$. Since the matrices $(c_{s_1 i}^{p_1})$ and $(c_{s_2 i}^{p_2})$ are not proportional, then it follows from (45) that two terms occurring in (45) vanish separately. In particular, this means that

$$c_{s_1 i}^{p_1} \theta_{q_2}^{s_1} \wedge \theta^{q_2} = 0. \tag{46}$$

Since the number of linearly independent forms among the 1-forms $\omega_i^{p_1}$ connected with the basis forms by relations (47) is equal to the number of linearly independent forms θ^{q_1} (i.e., it is equal r_1), then it follows from (45) that

$$\theta_{q_2}^{s_1} \wedge \theta^{q_2} = 0.$$

But the last equations coincide with equations (43) and are conditions of complete integrability of (42). Thus the manifold X is foliated into an r_1-parameter family of submanifolds of dimension $l+r_2$ and of rank r_2, and these submanifolds belong to the types described in Theorems 2 or 3.

In a similar way, one can prove the complete integrability of equations $\theta^{q_2} = 0$ on the manifold X. Thus the manifold X is foliated also into an r_2-parameter family of submanifolds of dimension $l + r_1$ and of rank r_1.

By induction over s, we can prove the result, which we have proved for $s = 2$ components, for the case of any number s of components. ■

Thus, Theorem 5 describes the structure of tangentially degenerate manifolds of general types. As a result, this theorem is a *structure theorem* for such manifolds.

Note that the torsal manifolds described in Theorem 2 are completely reducible, and the manifolds X described in Theorems 3 and 4 are irreducible manifolds.

Note that Theorem 5 does not cover tangentially degenerate submanifolds with multiple nonlinear components of their focal images. This gives rise to the following problem.

Problem. *Construct an example of a submanifold $X \subset P^N(\mathbf{C})$ with a degenerate Gauss map whose focal images have multiple nonlinear components or prove that such submanifolds do not exist.*

8 Additional results

In conclusion we prove two additional theorems.

Theorem 11 *Let $X \subset P^N$ be a tangentially degenerate submanifold of dimension n and rank $r < n$. Suppose that all matrices B^α can be simultaneously diagonalized, $B^\alpha = \mathrm{diag}\ (b^\alpha_{l+1,l+1}, \ldots, b^\alpha_{nn})$. Suppose also that the rectangular matrix B (defined by (20)) composed from the eigenvalues of the matrices B^α has a rank $r_1 \leq r-1$, and this rank does not reduce when we delete any column of this matrix. Then the submanifold X belongs to a subspace P^{n+r_1} of the space P^N.*

Proof. Under the conditions of Theorem 11, the second group of equations (11) takes the form

$$\omega^\alpha_p = b^\alpha_{pp}\theta^p, \ \ p = l+1, \ldots, n, \ \alpha = n+1, \ldots, N. \tag{47}$$

The matrix B has only r_1 linearly independent rows. Thus by means of transformations of moving frame's vertices located outside of the tangent subspace T_L, equations (47) can be reduced to the form

$$\omega^\lambda_p = b^\lambda_{pp}\theta^p, \ \ \omega^\sigma_p = 0, \tag{48}$$

where $\lambda = n+1, \ldots, n+r_1, \ \sigma = n+r_1+1, \ldots, N$. The second group of equations (9) takes the form

$$dA_p = \omega^i_p A_i + \omega^q_p A_q + \omega^\lambda_p A_\lambda,$$

and the points A_λ together with the points A_i and A_q define the osculating subspace S_L of the submanifold X for all points $x \in L$. The dimension of S_L is $n + r_1$, $\dim S_L = n + r_1$.

Differentiation of the points A_λ gives

$$dA_\lambda = \omega_\lambda^i A_i + \omega_\lambda^p A_p + \omega_\lambda^\mu A_\mu + \omega_\lambda^\sigma A_\sigma, \tag{49}$$

where $\lambda, \mu = n+1, \ldots, n+r_1;\ \sigma = n+r_1+1, \ldots, N$. If $\theta^p = 0$, then the osculating subspace S_L of X remains fixed. It follows from equations (49) that the 1-forms ω_λ^ρ are expressed in terms of the basis forms θ^p of X, that is,

$$\omega_\lambda^\rho = l_{\lambda p}^\rho \theta^p. \tag{50}$$

Taking exterior derivatives of the second group of equations (48), we find that

$$\omega_p^\lambda \wedge \omega_\lambda^\rho = 0. \tag{51}$$

Substituting the values of the 1-forms ω_p^λ and ω_λ^ρ from equations (48) and (50) into equation (51), we find that

$$b_{pp}^\lambda \theta^p \wedge l_{\lambda q}^\rho \theta^q = 0.$$

In this equation the summation is carried over the indices λ and q, but there is no summation over the index p. It follows from these equations that

$$b_{pp}^\lambda l_{\lambda q}^\rho = 0, \quad p \neq q. \tag{52}$$

System (52) is a system of linear homogeneous system with respect to the unknown variables $l_{\lambda q}^\rho$. For each pair of the values ρ and q, system (52) has the rank $r-1$ and r_1 unknowns. Since $r_1 \leq r-1$, under the conditions of Theorem 11, the rank of the matrix of coefficients of this system is equal r_1. As a result, the system has only the trivial solution $l_{\lambda q}^\rho = 0$. Thus equations (50) take the form

$$\omega_p^\lambda = 0. \tag{53}$$

It follows from (49) and (53) that the osculating subspace S_L of X remains fixed when L moves in X. Thus $X \subset P^{n+r_1}$. ■

Remark. If $r_1 = r$ and $N > n + r$, then the osculating subspace S_L of X can move in P^N when L moves in X. In this case the submanifold X is torsal.

Theorem 11 is similar to Theorem 3.10 from the book [AG 93] and was proved in [AG 93] for submanifolds of a space P^N bearing a net of conjugate lines. Note that Theorem 3.10 from [AG 93] generalizes a similar theorem of C. Segre (see [Se 07], p. 571) proved for submanifolds X of dimension n of the space P^N which has at each point $x \in X$ the osculating subspace S_x of dimension $n+1$. By this theorem, a submanifold X either belongs to a subspace P^{n+1} or is a torse.

The theorem dual to Theorem 11 is also valid.

Theorem 12 *Let $X \subset P^N$ be a tangentially degenerate submanifold of dimension n and rank $r < n$. Suppose that all matrices C_i can be simultaneously diagonalized, $C_i = \operatorname{diag}\,(c^{l+1}_{l+1,i}, \ldots, c^n_{ni})$. Suppose also that the rectangular matrix $C = (c^p_{pi})$ composed from the eigenvalues of the matrices C_i has a rank $r_2 \leq r-1$, and this rank is not reduced when we delete any column of this matrix. Then the submanifold X is a cone with an $(l-r_2)$-dimensional vertex K_L.*

Proof. The proof of this theorem is similar to the proof of Theorem 11. ■

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